

# Visualizing Spacetime Curvature via Gradient Flows I: Introduction

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Traditional approaches to the study of the dynamics of spacetime curvature in a very real sense hide the intricacies of the nonlinear regime. Whether it be huge formulae, or mountains of numerical data, standard methods of presentation make little use of our remarkable skill, as humans, at pattern recognition. Here we introduce a new approach to the visualization of spacetime curvature. We examine the flows associated with the gradient fields of invariants derived from the spacetime. These flows reveal a remarkably rich structure, and offer fresh insights even for well known analytical solutions to Einstein's equations. This paper serves as an overview and as an introduction to this approach.

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## I. INTRODUCTION

To quote from a recent monograph [1]: “*It appears that it is much easier to find a new solution of Einstein's equations than it is to understand it*”. The present work involves the novel use of both computer algebra (for background calculations) and numerical routines (for visualization) with the objective being the development of a fresh view of “curvature”. In this endeavor we are not alone. In [2] a visualization of spacetime curvature based on the electric and magnetic components of the Weyl tensor is being developed. In [3] a potentially complementary approach is also under development. Here we examine something akin to a dynamical systems approach: we look at the “flows” associated with the gradient fields of invariants. We show that these flows reveal a remarkably rich structure and fresh insights even for well known geometries. The present approach does not single out any particular observer, nor theory, but is designed to reflect deep intrinsic properties of the background geometry itself.

## II. GRADIENT VECTOR FLOWS

### A. Invariants

Let  $(M, g)$  be a semi - Riemannian manifold of dimension  $d$ <sup>1</sup> where  $g$  is the metric tensor. The simplest construction of invariants involves scalars formed from metric contractions and partial derivatives to order  $p$  (scalars polynomial in the Riemann tensor). The number ( $\mathcal{N}$ ) of algebraically (not functionally) independent scalars of this type (invariants not satisfying polynomial degenera-

cies (syzygies)) is given by [4]

$$\mathcal{N}(d, p) = \frac{d(d+1)((d+p)!)}{2d!p!} - \frac{(d+p+1)!}{(d-1)!(p+1)!} + d \quad (1)$$

for  $d > 2, p \geq 2$  and so, for example, in spacetime the number of non-differential invariants (no derivatives of the Riemann tensor) is  $\mathcal{N}(4, 2) = 14$ . This does not mean that a single set of 14 invariants will cover all possibilities. The situation is rather more involved [5] and the issue of a minimal complete set remains unresolved even at  $d = 4$ . Physical situations involve many simplifications, and a consequent reduction in  $\mathcal{N}$ . For example, it is well known that in the Ricci - flat case there are at most 4 invariants at order  $p = 2$ . Moreover, some invariants can be distinguished when their physical meaning is clear<sup>2</sup> and some can be used to construct Newtonian “analogues” as explained below. Whereas the mathematics of polynomial invariants remains a very active area of research (see, for example, [6] and references therein), as recently pointed out by Page [7], non-polynomial invariants can be constructed even when all polynomial invariants vanish.

### B. Vector Fields

Given  $(M, g)$  and a set of invariants  $\mathcal{I}_n$  (we need not specify  $d$  nor  $p$  at this point nor even the nature of the  $\mathcal{I}_n$ ) consider the gradient flows<sup>3</sup> [8]

$$k_n^\alpha \equiv \pm \nabla^\alpha \mathcal{I}_n = \pm g^{\alpha\beta} \frac{\partial \mathcal{I}_n}{\partial x^\beta}. \quad (2)$$

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<sup>1</sup> We are mainly concerned here with  $d = 4$  but higher dimensions can be easily included

<sup>2</sup> For example, in Einstein's theory in 4-dimensions, without cosmological constant and for a perfect fluid,  $\mathcal{R} = 8\pi(\rho - 3p)$ , where  $\mathcal{R}$  is the Ricci scalar,  $\rho$  the energy density and  $p$  the isotropic pressure.

<sup>3</sup> We refer to the choice “+” as a positive gradient flow and the choice “-” as a negative gradient flow. We find occasion to use both.

Note that  $k$  can be timelike, spacelike or null. Also note that even if a flow resides entirely in a subspace, calculations must be carried out in the full space since invariants of the full space are not, of course, the same as invariants of subspaces. For notational simplicity we now consider  $n$  understood. By construction, we have

$$\nabla_{[\alpha} k_{\beta]} = k_{[\alpha} \nabla_{\beta} k_{\gamma]} = 0 \quad (3)$$

and so on. In terms of the norm

$$k_{\alpha} k^{\alpha} \equiv \mathcal{V}, \quad (4)$$

it follows from (3) that the acceleration associated with  $k$  is<sup>4</sup>

$$k^{\beta} \nabla_{\beta} k_{\alpha} = \frac{1}{2} \nabla_{\alpha} \mathcal{V}. \quad (5)$$

From (2) it follows that the associated expansion is

$$\nabla_{\alpha} k^{\alpha} = \pm \nabla_{\alpha} (\nabla^{\alpha} \mathcal{I}) \equiv \pm \square \mathcal{I}. \quad (6)$$

In terms of the projection tensor  $h_{\beta}^{\alpha} \equiv \delta_{\beta}^{\alpha} \pm k^{\alpha} k_{\beta}$ , borrowing the usual definition of the vorticity tensor  $\omega_{\alpha\beta} \equiv h_{\alpha}^{\gamma} h_{\beta}^{\delta} \nabla_{[\gamma} k_{\delta]}$ , it follows immediately that

$$\omega_{\alpha\beta} = 0. \quad (7)$$

### C. Special Symmetries

For any scalar field  $\mathcal{I}$ , the Lie derivative associated with a vector field  $\xi$  is given by [9]

$$\mathcal{L}_{\xi} \mathcal{I} = \xi_{\alpha} \nabla^{\alpha} \mathcal{I} = \xi_{\alpha} k^{\alpha}. \quad (8)$$

In certain circumstances, the scalar  $\xi_{\alpha} k^{\alpha}$  can be directly related to the invariant  $\mathcal{I}$  itself. For example, if the manifold admits a homothetic motion,  $\mathcal{L}_{\xi} g_{\alpha\beta} = 2\Phi g_{\alpha\beta}$ , where  $\Phi (= \nabla_{\alpha} \xi^{\alpha} / d)$  is constant, it is known that for polynomial invariants [10]

$$\mathcal{L}_{\xi} \mathcal{I} = \kappa \Phi \mathcal{I} \quad (9)$$

where  $\kappa$  is an integer characteristic of  $\mathcal{I}$  (involving, for example,  $p$  and the number of discrete contractions used to make  $\mathcal{I}$ ). In the case that  $\xi$  is Killing ( $\Phi = 0$ ) we have

$$\xi_{\alpha} k^{\alpha} = 0. \quad (10)$$

That is, polynomial gradient flows are orthogonal to Killing flows.<sup>5</sup> This result can be used to clarify and extend the classical notions of  $\mathcal{R}$  and  $\mathcal{T}$  regions of spacetime. This is explored in Appendix A.

### D. Special Invariants and Newtonian Analogues

As mentioned above, in physical situations there is a reduction in  $\mathcal{N}$  due to simplifying assumptions. For example, in class  $B$  warped product spacetimes (which includes, for example, all spherical, plane and hyperbolic spacetimes) it is known that  $\mathcal{N}(4, 2) \leq 4$  and that  $\mathcal{N}(4, 2) = 1$  in the Ricci-flat case [21]. However, in addition to simplifications leading to a decrease in the number of invariants that need to be considered, it has to be realized that not all invariants are equally useful. In this section, special invariants, that allow the construction of Newtonian analogues, are examined.<sup>6</sup>

Decompose the Weyl tensor into its electric and magnetic parts in the usual way: For a unit timelike vector  $u^{\alpha}$ ,  $E_{\alpha\gamma} \equiv C_{\alpha\beta\gamma\delta} u^{\beta} u^{\delta}$  and  $H_{\alpha\gamma} \equiv \tilde{C}_{\alpha\beta\gamma\delta} u^{\beta} u^{\delta}$  where  $\tilde{C}$  is dual to the Weyl tensor  $C$ . Since  $H_{\alpha\beta}$  has no Newtonian counterpart, consider here the simplest case, purely electric spacetimes  $H_{\alpha\beta} = 0$  (e.g. all static spacetimes, shear-free and hypersurface orthogonal perfect fluid spacetimes *etc.*). Following for example Ellis [22], construct the Newtonian tidal tensor

$$E_{ab} = \Phi_{,a,b} - \frac{1}{3} \eta_{ab} \square \Phi \quad (11)$$

and define the associated invariant

$$E_{ab} E^{ab} = \Phi_{,a,b} \Phi^{,a,b} - \frac{1}{3} (\square \Phi)^2. \quad (12)$$

We now construct two gradient fields

$$k^{\gamma} \equiv \pm \nabla^{\gamma} (E_{\alpha\beta} E^{\alpha\beta}), \quad l^c \equiv \pm \nabla^c (E_{ab} E^{ab}). \quad (13)$$

We say that  $l$  is a Newtonian analogue (in no way any limit) of  $k$  if their associated phase portraits are “analogous”. By “analogous” we refer to, for example, the topology of the flow.<sup>7</sup> To further clarify this construction, let us recall that the tidal tensor in standard general relativity is given by [23]  $\mathcal{T}_{\alpha\gamma} \equiv \mathcal{R}_{\alpha\beta\gamma\delta} u^{\beta} u^{\delta}$  where again  $u^{\alpha}$  is a unit timelike vector and  $\mathcal{R}$  is the Riemann tensor. In the Ricci - flat case then  $\mathcal{T}_{\alpha\gamma} = E_{\alpha\gamma}$  and since  $C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} \propto E_{\alpha\beta} E^{\alpha\beta}$  [24] the analogy constructed is strict since  $C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} \in \mathcal{I}$ . Now, whereas  $\nabla^{\gamma}$  is constructed in  $(M, g)$ ,  $\nabla^c$  is constructed in a Euclidean 3-space with metric  $\eta_{ab}$ . Clearly the use of the term “analogous” must account for this. The Curzon - Chazy solution provides a detailed demonstration, and clarification, of these ideas [25].

### E. Singularities

The gradient flows (2) can encounter a variety of exceptional situations, the most common of which are critical

<sup>4</sup> It is essential that  $k$  is not normalized, since with (5), normalization would restrict  $k$  to a geodesic flow.

<sup>5</sup> In the special circumstance that  $k$  itself satisfies  $\mathcal{L}_k g_{\alpha\beta} = 2\Phi g_{\alpha\beta}$ , it follows that  $k$  is concurrent [9],  $\nabla_{\alpha} k_{\beta} = \Phi g_{\alpha\beta}$ , and so the associated streamlines are geodesic with  $\nabla_{\alpha} \mathcal{V} = 2\Phi k_{\alpha}$ .

<sup>6</sup> A much simpler “analogue” is to simply construct the associated potential flow, the potential being the Newtonian potential. This is discussed below.

<sup>7</sup> Of course, the flow must have sufficient structure so as to be able to form an “analogue”.

points were  $k^\alpha = 0$ . This is discussed below. Perhaps the simplest exceptional situation is the development of a caustic where the expansion of the flow diverges so that from (6)

$$|\Box \mathcal{I}| \rightarrow \infty. \quad (14)$$

In a recent influential work [26], it has been claimed that attempts to explain cosmic acceleration, by way of observations in locally inhomogeneous spacetimes, necessarily involves a local “weak singularity” at the origin. This argument is based on the observation that  $\Box \mathcal{R}$  necessarily diverges at the origin in these models. This point of view has been criticized [27]. In terms of a gradient flow, as introduced here, the correct way to view this controversy, is to observe that for  $\mathcal{I} = \mathcal{R}$  this “weak singularity” is actually a caustic in the gradient flow of  $\mathcal{R}$ , as follows from (6). Since the models under consideration are simply dust,  $k$  represents the gradient flow of the energy density and the “weak singularity” derives from the lack of sufficiently high differentiability of the energy density at the origin. More substantial, and certainly more well known singularities, occur where an  $\mathcal{I}$  diverges (so called “scalar polynomial singularities” in the simplest cases). In this circumstance, the “singularity” must be expanded to reveal the structure of the associated gradient flow about the “singularity”. In archetypical cases this reveals a very rich structure [25] [28].

### III. DYNAMICAL SYSTEMS

#### A. Contact With Physics

Let us assume regular coordinates  $x^\alpha$  in a region of  $(M, g)$  and write the negative gradient flow in the form [29]

$$k^\alpha = \dot{x}^\alpha = \mp g^{\alpha\beta} \frac{\partial \mathcal{I}}{\partial x^\beta} \quad (15)$$

where  $\cdot \equiv d/d\lambda$  and  $\lambda$  is any natural parametrization of the curve  $x^\alpha(\lambda)$  with tangent  $k^\alpha$ . It is clear that we are dealing with an autonomous dynamical system. In order to make an immediate contact with physics, we simply note here that the character of critical points associated with the system (15) for the Kerr metric is dependent on the value of the rotational parameter  $A$  [30]. There is “hidden” information intrinsic to the Kerr metric not evident without the study of the associated gradient flows.

#### B. Critical Points

The phase portrait associated with (15) is dominated by critical points, the deep study of which leads to Morse Theory [31]. Gradient systems, in the large, have many diverse uses, and are the subject of much modern mathematical study. Our considerations here involve gradient

flows on  $(M, g)$ , yet most information readily available involves gradient flows on  $E^n$ , Euclidean  $n$ -space. This situation is addressed in Appendix B. In Appendix C we characterize fixed points by indices.

## IV. ELEMENTARY EXAMPLES

### A. The Robertson-Walker Spacetime

Perhaps the most widely recognized spacetime is that of Robertson and Walker,

$$ds^2 = a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega_2^2 \right) - dt^2. \quad (16)$$

Due to the isotropy, it is clear, *a priori*, that the associated gradient “flows” can only be 1-dimensional. There can be no isolated critical points. The positive gradient flows are given by

$$k^\alpha = - \frac{d\mathcal{I}}{dt} \delta_t^\alpha \quad (17)$$

so that

$$\mathcal{V} = - \left( \frac{d\mathcal{I}}{dt} \right)^2 \leq 0. \quad (18)$$

The flows are timelike except at critical isotropic 3-surfaces for which  $d\mathcal{I}/dt = 0$ . There are exactly two independent invariants at order  $p = 2$ , Ricci invariants, and allowing rational exponents of polynomial invariants, these invariants can be taken to be simply the energy density  $\rho$  and isotropic pressure  $p$ . Turn - around Universes, those for which  $da/dt|_{t_0} = 0$ , define (irrespective of any cosmological constant) critical 3-surfaces  $t = t_0$ .

### B. The Schwarzschild Spacetime

Consider the Schwarzschild vacuum in dimensionless null coordinates  $(u, v, \theta, \phi)$ ,

$$ds^2 = 4m^2 \left( \frac{4\mathcal{L}}{(1+\mathcal{L})} \frac{dudv}{uv} + (1+\mathcal{L})^2 d\Omega_2^2 \right), \quad (19)$$

where  $m > 0$  is the effective gravitational mass,  $\mathcal{L} \equiv \mathcal{L}(uv)$  and  $\mathcal{L}$  is the Lambert W function [34] [35]. It remains convenient to define  $r \equiv 2m(1+\mathcal{L})$ . It follows that  $\xi^\alpha \equiv (u, -v, 0, 0)$  is a Killing vector (and tangent to trajectories of constant  $r$ ). We find

$$\xi^\alpha \xi_\alpha = \frac{(4m)^2}{r} (2m - r), \xi^\alpha \nabla_\alpha \xi^\beta = \left( \frac{2m}{r} \right)^2 (u, v, 0, 0) \quad (20)$$

and so along  $u = 0$  or  $v = 0$  ( $r = 2m$ ),  $\xi$  is tangent to a radial null geodesic. Further, we find that  $\nabla_\alpha \xi^\alpha = \xi_{[\alpha} \nabla_\beta \xi_{\gamma]} = 0$  so that regions for which  $r > 2m$  are static; all of which is, of course, well known.

For (19), there is only one independent invariant of order  $p = 2$ , and we can take this to be the square of the Weyl tensor

$$\mathcal{I} = \frac{3}{4m^4(1+\mathcal{L})^6}. \quad (21)$$

It follows that the tangents to the negative gradient flow are given by  $k^\alpha \propto -m(u, v, 0, 0)/r^7$  (tangent to trajectories of constant “Schwarzschild  $t$ ”),

$$\mathcal{V} \propto \frac{m^4}{r^{15}}(r - 2m), k^\alpha \nabla_\alpha k^\beta \propto \frac{(15m - 7r)m^3}{r^{16}}(u, v, 0, 0), \quad (22)$$

and  $\nabla_\alpha k^\alpha \propto m^2(5r - 12m)/r^9$ . Again, along  $u = 0$  or  $v = 0$ ,  $k$  is tangent to a radial null geodesic. The  $\mathcal{R}$  region corresponds to  $r > 2m$  whereas the  $\mathcal{T}$  region corresponds to  $r < 2m$ , exactly as expected. Both  $\xi$  and  $k$  vanish identically at the isolated critical point  $\mathcal{P}$ , the bifurcation 2-sphere  $u = v = 0$  [36].

Let us now be more explicit about the flow with the aide of Appendices B and C. We find

$$k^u = \dot{u} = \psi u, \quad k^v = \dot{v} = \psi v \quad (23)$$

where

$$\psi \equiv -\left(\frac{3}{4m^3}\right)^2 \frac{1}{(1+\mathcal{L})^7}. \quad (24)$$

In particular, we find

$$\mathcal{D}|_{\mathcal{P}} = \left(\frac{3m^3}{4}\right)^4 > 0 \quad (25)$$

and

$$\left.\frac{\partial G}{\partial x^1}\right|_{\mathcal{P}} = \left(\frac{3}{4m^3}\right)^2 > 0 \quad (26)$$

and so the isolated critical point is a stable node and the associated index is  $+1$ .<sup>8</sup> In the neighborhood of  $uv = 0$ , it follows from (24) that  $\psi \sim -(3/4m^3)^2$  and so from (23) it follows that we have a simple linear product flow in the neighborhood of  $uv = 0$ . Such flows have a natural interpretation in terms of gradient flows [32].

The phase portrait in this case is very simple and the associated Newtonian potential flow is easily constructed. Setting the center of symmetry at the bifurcation two-sphere, and using Cartesian coordinates, it follows that the associated Newtonian potential flow is that of a fluid sphere of uniform density. In situations with more substantial phase portraits, one can go beyond associated potential flows and construct Newtonian analogues as explained above.

### C. Weyl Invariants and Spherically Symmetric Spacetime

As mentioned above, in class  $B$  warped product spacetimes it is known that  $\mathcal{N}(4, 2) \leq 4$  and that  $\mathcal{N}(4, 2) = 1$  in the Ricci-flat case [21]. Here we examine Weyl invariants within spherical symmetry. (In the notation of [21], we need only consider the first Weyl invariant  $w_1$  (equivalent to the square of the Weyl tensor, divided by 8).) Now *locally*, every spherically symmetric spacetime can be written in the form

$$ds^2 = -2f(u, v)dudv + r(u, v)^2 d\Omega_2^2. \quad (27)$$

It follows that the negative gradient flow of  $w_1$  is given by

$$k^\alpha = -\Delta(u, v)(\mu(u, v), \delta(u, v), 0, 0) \quad (28)$$

with

$$\mathcal{V} \propto -\Delta^2 f \mu \delta \quad (29)$$

where

$$w_1 \propto \Delta^2 f^{10} r^6 \quad (30)$$

and

$$\Delta \equiv \frac{\Psi(u, v)}{3f^8 r^5}. \quad (31)$$

$\Psi$ ,  $\mu$  and  $\delta$  involve derivatives of  $f$  and  $r$  up to order  $p = 2$ . The explicit forms are not needed here. Rather, we note that there exist two distinct types of critical points associated with the gradient flows of  $w_1$ . From (28) and (31) it follows that the critical points can be characterized by the *local* conditions

$$\Psi = 0, \quad \mu = \delta = 0. \quad (32)$$

In the Robertson-Walker case, of course,  $\Psi = 0$  is a global condition. The Schwarzschild spacetime demonstrates a case where  $\Psi \neq 0$  locally. Rather, the bifurcation 2-sphere corresponds to an isolated critical point characterized by  $\mu|_{\mathcal{P}} = \delta|_{\mathcal{P}} = 0$ . We call such critical points locally *anisotropic*.

In order to *explicitly* demonstrate examples where  $\Psi = 0$  locally, consider, for example, static perfect fluids in comoving coordinates,

$$ds^2 = \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 d\Omega_2^2 - e^{\Phi(r)} dt^2. \quad (33)$$

Given  $\Phi$  sufficiently smooth, and subject to boundary conditions, one can generate the effective gravitational mass  $m$  that gives rise to a perfect fluid [37]. For example, setting

$$\Phi = \frac{1}{2} N \ln(1 + \frac{r^2}{\alpha}) \quad (34)$$

<sup>8</sup> As mentioned in Appendix A, the introduction of a cosmological constant  $\Lambda$  alters the topology of the conformal boundary of spacetime. In particular, the Schwarzschild de Sitter case has two bifurcate 2-spheres and a consequent index of  $+2$ .

where  $N$  is an integer  $\geq 1$  and  $\alpha$  is a constant  $> 0$ , an infinite number of exact perfect fluid solutions follow. For all such models we have

$$w_1 = \mathcal{V} = \square w_1 = 0 \quad (35)$$

at the center of symmetry  $r = 0$ . Moreover,  $k^\alpha = (0, 0, 0, 0)$  there and so we have a critical 3-surface at the origin based on the Weyl invariant. That is, the solutions are locally isotropic about the origin for all  $t$ . We call such a collection of critical points locally *isotropic*.

## V. DISCUSSION

This paper serves as an introduction to the study of gradient flows of invariants as a means to visualize curvature. No particular observer, or theory, is fundamental to this approach, but the intention is to apply the techniques to spacetimes associated with Einstein's theory of gravity. It has been shown that gradient flows of invariants polynomial in the Riemann tensor are necessarily orthogonal to Killing flows, should they exist. This allows for a rigorous definition and generalization of the classical notions of  $\mathcal{R}$  and  $\mathcal{T}$  regions of spacetime. For purely electric spacetimes one can construct strict Newtonian analogues, based on the topology of flows; for the first Weyl (tidal) invariant in spacetime, and for the tidal invariant in Newtonian theory. This is discussed at length elsewhere [25].

A central feature of many spacetimes is the development of “singularities”, a concept which remains open

to much refinement. It has long been known [38] that naked spherically symmetric singularities are “massless”, a fact clearly reflected in the associated gradient flows [28]. This invites a reexamination of the very notion of “singularity”.

From the point of view of dynamical systems, gradient flows are simple in the sense that critical points are the only possible limit sets of the flow. We have translated standard results in  $E^n$  into the setting of a semi-Riemannian manifold and reviewed the importance of the index associated with each critical point, reflecting on the role of the cosmological constant as regards its role on the fundamental index of a spacetime.

We have reviewed examples, restricted here to spherical symmetry. A rather complete analysis of the Kruskal - Szekeres vacuum was given, interpreting the associated bifurcate 2-sphere as the isolated critical point of the solution. More generally, isolated critical points of the Weyl invariant within spherical symmetry were distinguished as locally isotropic or anisotropic, with explicit examples given for each.

Finally, we note that even at the level of complexity offered by the Kerr spacetime, a remarkably rich structure is revealed by the associated gradient field [30].

## ACKNOWLEDGMENTS

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- [28] Majd Abdelqader, James Wurster and Kayll Lake, "*Visualizing Spacetime Curvature via Gradient Flows III: Archetypical Naked Singularities*" (in preparation).
- [29] A negative sign is introduced here as this is the most common form of a gradient flow one sees in the dynamical systems approach. This has no physical consequences.
- [30] Majd Abdelqader, James Wurster and Kayll Lake, "*Visualizing Spacetime Curvature via Gradient Flows IV: The Remarkable Structure of the Kerr Metric*" (in preparation).
- [31] See, for example, J. Jost, *Riemannian Geometry and Geometric Analysis* Sixth Edition (Springer, New York, 2011).
- [32] See, for example, J. Hale and H. Ko  ak, *Dynamics and Bifurcations* (Springer-Verlag, New York, 1991).
- [33] For a readable account see, for example, L. Perko, *Differential Equations and Dynamical Systems* (Springer-Verlag, New York, 1991).
- [34] This is defined by  $\mathcal{L}(x)e^{\mathcal{L}(x)} = x$ . See, for example, R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, *Advances in Computational Mathematics* **5**, 329 (1996).
- [35] Note that our simple labeling of  $u$  and  $v$  differs from the usual labeling which uses  $\mathcal{L}(-uv/e)$ .
- [36] The use of a complete covering of the manifold *ab initio* here is crucial as regards isolated critical points. If, for example, one was to use traditional "Schwarzschild" coordinates  $(r, \theta, \phi, t)$  then the span  $(r = 2m, -\infty < t < \infty)$  is critical in the sense that  $k^\alpha = \xi_\alpha = 0$  over this span which is, of course, but a subset of the bifurcation two-sphere.
- [37] K. Lake, *Phys. Rev. D* **67**, 104015 (2003) (gr-qc/0209104).
- [38] K. Lake, *Phys. Rev. Lett.* **68**, 3129 (1992).
- [39] This package runs within Maple. The GRTensorII software and documentation is distributed freely from the address <http://grtensor.org>

## Appendix A: $\mathcal{R}$ and $\mathcal{T}$ regions

The gradient of the areal radius of an arbitrary spherically symmetric field can be spacelike, timelike or null. A careful distinction of these possibilities is important and, for example, forms an essential element of any complete proof of the Birkhoff theorem (for example [11]). The possibility of spacelike and timelike gradients was studied extensively in the Russian literature (and labeled " $\mathcal{R}$ " and " $\mathcal{T}$ " regions respectively) some forty years ago [12] and yet there appears to be no readily available extension of these ideas. It is the purpose of this Appendix to explore this using gradient flows.

The original distinction of  $\mathcal{R}$  and  $\mathcal{T}$  regions of spacetime was restricted to spherical symmetry and given by Novikov [12] (see also the work of Ruban [12]) by way of a coordinate construction [13]. This construction can be made invariant as follows: Consider a spherically symmetric spacetime [14]

$$ds^2 = ds_\Sigma^2 + R^2 d\Omega_2^2 \quad (\text{A1})$$

where  $d\Omega_2^2$  is the metric of a unit 2-sphere ( $d\theta^2 + \sin^2(\theta)d\phi^2$ ) and  $R = R(x^1, x^2)$  (the areal radius) where the coordinates on the Lorentzian 2-space  $\Sigma$  are labeled as  $x^1$  and  $x^2$ . (No specific choice of coordinates on  $\Sigma$  need be made.) Writing the effective gravitational mass (sometimes called the Misner-Sharp energy) as  $m = m(x^1, x^2)$  [15] it follows from Novikov's coordinate definitions that

$$R > 2m, \quad R = 2m, \quad R < 2m \quad (\text{A2})$$

in an  $\mathcal{R}$  region, on the boundary and in a  $\mathcal{T}$  region respectively [16]. In more modern notation [17], an  $\mathcal{R}$  region is untrapped, a  $\mathcal{T}$  region trapped and the boundary marginal. Whereas these distinctions are fundamental, they are restricted to strict spherical symmetry. The natural extension of these ideas beyond spherical symmetry involves finding the apparent horizon(s) [11], a key element of numerical relativity [18].

In order to bring gradient flows into the picture, let us note that a stationary region of spacetime admits a time-like Killing congruence. Since every non-zero 4-vector orthogonal to a timelike 4-vector must be spacelike, it follows from (10) that any gradient flow is necessarily spacelike in a stationary region. From (4) we have  $\mathcal{V} > 0$

throughout an  $\mathcal{R}$  region, a fact that we can use to define such a region. Now let us assume that there is a region in which the gradient flow is timelike so that  $\mathcal{V} < 0$ . (A Killing congruence in this region, should one exist, is then necessarily spacelike.) We can use  $\mathcal{V} = 0$  to define a  $\mathcal{T}$  region. Boundary regions are then naturally defined by  $\mathcal{V} = 0$ . The foregoing classification is independent of symmetries and we adopt this classification in general:  $\mathcal{V} > 0$  throughout an  $\mathcal{R}$  region where  $k$  is spacelike,  $\mathcal{V} < 0$  throughout an  $\mathcal{T}$  region where  $k$  is timelike, and  $\mathcal{V} = 0$  on a boundary where  $k$  is null or the zero vector. It is already known [19], for example, that for the Kerr metric  $\mathcal{V} \geq 0$  for the rotational parameter  $A \geq 1$  with equality holding only in the degenerate case  $A = 1$ . That is, degenerate and naked Kerr metrics do not admit  $\mathcal{T}$  regions. Such statements are, however, strongly dependent on the sign of the cosmological constant ( $\Lambda$ ). If  $\Lambda > 0$  then conformal infinity is spacelike [20] and so one would expect that the degenerate and naked cases would then admit no  $\mathcal{R}$  regions.

## Appendix B: The Euclidean Counterpart

In this Appendix we wish to convert the covariant statement

$$\dot{x}^\alpha = \mp g^{\alpha\beta} \frac{\partial \mathcal{I}}{\partial x^\beta}$$

into the non-covariant but common form

$$\dot{x}^\alpha = \mp \frac{\partial F}{\partial x^\alpha} \quad (\text{B1})$$

for the purpose of exploiting many well-known properties of the system (B1). Here we restrict our considerations to 2-dimensional flows, but we note that higher dimensional cases follow analogously. From (15) we have

$$(\dot{x}^1, \dot{x}^2, 0 \dots) = \mp (g^{11} \frac{\partial \mathcal{I}}{\partial x^1} + g^{12} \frac{\partial \mathcal{I}}{\partial x^2}, g^{21} \frac{\partial \mathcal{I}}{\partial x^1} + g^{22} \frac{\partial \mathcal{I}}{\partial x^2}, 0 \dots). \quad (\text{B2})$$

We now simply write out the associated system

$$(\dot{x}^1, \dot{x}^2) = \mp \left( \frac{\partial F}{\partial x^1}, \frac{\partial F}{\partial x^2} \right) \equiv (G, H) \quad (\text{B3})$$

where the identifications for  $F$  are obvious, and the last equivalence is for notational convenience. Note that we never need solve for  $F$ . Rather, we proceed in a purely mechanical way to construct the Hessian and its discriminant  $\mathcal{D}$ . We need not write these out explicitly here. A critical point  $\mathcal{P}$ , where  $\dot{x}^1 = \dot{x}^2 = 0$ , is degenerate

if the discriminant vanishes,  $\mathcal{D}|_{\mathcal{P}} = 0$ . Otherwise  $\mathcal{P}$  is non-degenerate, that is, a Morse critical point. We are now in a position to summarize the situation [32]: Morse critical points of a gradient flow are hyperbolic. Further, for isolated Morse critical points we have:

- $\mathcal{P}$  is an unstable node if and only if  $\mathcal{D}|_{\mathcal{P}} > 0$  and  $\partial G / \partial x^1|_{\mathcal{P}} < 0$ .
- $\mathcal{P}$  is an asymptotically stable node if and only if  $\mathcal{D}|_{\mathcal{P}} > 0$  and  $\partial G / \partial x^1|_{\mathcal{P}} > 0$ .
- $\mathcal{P}$  is a saddle point if and only if  $\mathcal{D}|_{\mathcal{P}} < 0$ .

It is to be noted that gradient flows are simple flows in the sense that critical points are the only possible limit sets. However, since we must also deal with singularities in the invariants  $\mathcal{I}$ , we must also deal with sources and sinks where one or more of the  $\dot{x}^\alpha$  diverge.

Now whereas we do not *have* to solve for  $F$ , we can attempt to do so. In particular, we can write

$$\dot{x}^\alpha = \mp \frac{\partial \Phi}{\partial x^\alpha}. \quad (\text{B4})$$

If we take  $\Phi$  to be the Newtonian potential, we refer to the flow (B4) as the associated Newtonian potential flow.

## Appendix C: Index Theory

Again here we restrict our considerations to 2-dimensional flows, higher dimensional cases follow analogously. We wish to characterize the topology of the 2-surface associated with the flow [33]. We note the following regarding the Poincaré index:

- The index of a node is +1.
- The index of a source or sink<sup>9</sup> is +1.
- The index of a hyperbolic saddle point is -1.
- The index of a closed curve containing fixed points is equal to the sum of the indices of the fixed points within.

The importance of calculating these indices follows from the Poincaré - Hopf theorem which we state here in the following simple form: Suppose the flow is on a 2-surface  $\Sigma$ . Calculate the sum of the indices of all fixed points on  $\Sigma$  with a suitable curve  $\gamma$  which contains all isolated fixed points. Then this sum is the Euler Characteristic of  $\Sigma$  within  $\gamma$ .

<sup>9</sup> These are included so as to be able to cover singular fixed points

where one or more of the  $\dot{x}^\alpha$  diverge.